Lecture 23 : Stopping Times and Martingales: Examples

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These notes are the continuation of the notes scribed by Moorea Brega. References: [1], sections 3.1 and 4.2.

23.1 Stopping Times and Martingales: Examples

Example 23.1 (Unfair coin tossing game) *Suppose that we have a biased coin, with probability p of heads, $q = 1 - p$ of tails. Let us define i.i.d. random variables X_i by $X_i = 1$ when the i^{th} coin toss is a head, and -1 when the i^{th} coin toss is a tail. Suppose that $S_0 = a$ where a is a positive integer, and let $S_n = S_0 + X_1 + \ldots + X_n$. Let $T = \inf\{n : S_n = 0 \text{ or } b\}$ for $b > 0$ an integer. We argue that $\mathbb{P}(T < \infty) = 1$ and we want to find $\mathbb{P}(S_T = b)$ and $\mathbb{P}(S_T = 0)$.*

Proof: Note that S_n is *not* a martingale and so the idea is to find a suitable $h(x)$ such that $h(S_n)$ is a martingale. If $h(S_n)$ is to be a martingale, we must have that $h(x) = ph(x+1) + qh(x-1)$ because given S_n, S_{n+1} is $S_n + 1$ with probability p and $S_n - 1$ with probability q. Consider $h(x) = z^x$ where z will be determined shortly. From the above equation, we have $z^x = pz^{x+1} + qz^{x-1}$ and in particular, $z = pz^2 + q$. Alternatively, compute $\mathbb{E}[z^{S_{n+1}}|\mathcal{F}_n] = z^{S_n}\mathbb{E}[z^{X_{n+1}}|\mathcal{F}_n] = z^{S_n}[pz^1 + qz^{-1}]$. If z^{S_n} is a martingale, $\mathbb{E}[z^{S_{n+1}}|\mathcal{F}_n] = z^{S_n}$ and so $z = pz^2 + q$. The roots of the quadratic equations are 1 (trivial) and (q/p). So we conclude that $(q/p)^{S_n}$ is a martingale.

For $0 \le n \le T = \inf\{n : S_n = 0 \text{ or } b\}$, we observe that $(q/p)^{S_n}$ is bounded between $(q/p)^0$ and $(q/p)^b$, so $(q/p)^{S_{n\wedge T}}$ is a bounded martingale. Easily, $\mathbb{P}(T < \infty) = 1$ and as $n \to \infty$, $S_{n\wedge T} \to S_T$, so $(q/p)^{S_{n\wedge T}} \to (q/p)^{S_T}$ which implies that $\mathbb{E}[(q/p)^{S_T}] = (q/p)^a$, where $S_0 = a$. Therefore,

$$\mathbb{P}(S_T = b)(q/p)^b + \mathbb{P}(S_T = 0)(q/p)^0 = (q/p)^a.$$

We also have

$$\mathbb{P}(S_T = b) + \mathbb{P}(S_T = 0) = 1.$$

Solving these two equations, we obtain

$$\mathbb{P}(S_T = b) = \frac{(q/p)^a - 1}{(q/p)^b - 1}$$

and

$$\mathbb{P}(S_T = 0) = \frac{(q/p)^b - (q/p)^a}{(q/p)^b - 1}.$$

Note that the two equations are linearly dependent for $p = q$ but we have already worked out the formula for this case in the previous lecture. ■

Example 23.2 (Wald's second equation) *Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{E}X_n = 0$ and $\mathbb{E}X_n^2 = \sigma < \infty$. If T is a stopping time with $\mathbb{E}T < \infty$ then show that $\mathbb{E}S_T^2 = \sigma^2\mathbb{E}T$.*

Proof: Recall that $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Let $M_n := S_n^2 - n\sigma^2$, where $\sigma^2 = \mathbb{E}[X^2]$. Check that M_n is a martingale:

$$\begin{aligned}
\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}[S_{n+1}^2 - (n+1)\sigma^2|\mathcal{F}_n] \\
&= \mathbb{E}[S_n^2 + 2X_{n+1}S_n + X_{n+1}^2 - (n+1)\sigma^2|\mathcal{F}_n] \\
&= S_n^2 - n\sigma^2 \\
&= M_n.
\end{aligned}$$

First consider the case of a bounded stopping time T. We know $0 = \mathbb{E}[M_0] = \mathbb{E}[M_T] = \mathbb{E}[S_T^2 - T\sigma^2]$. Therefore, $\mathbb{E}[S_T^2] = \mathbb{E}[T]\sigma^2$ if $\mathbb{P}(T \leq N) = 1$ for some non-random $N < \infty$.

We now look at the general case when T is unbounded. Consider $T \wedge n$ instead of T. We have $\mathbb{E}[S_{T\wedge n}^2] = \mathbb{E}[T \wedge n]\sigma^2$ for every $n = 1, 2, 3, \ldots$. Let $n \to \infty$. Since $T \wedge n \uparrow T$ as n increases, $\mathbb{E}[T \wedge n] \uparrow \mathbb{E}[T] < \infty$ (by assumption). Also we know that $S_{T\wedge n}$ is a martingale (because $S_n, n = 0, 1, \ldots$ is a martingale). However, since a martingale in L^2 has orthogonal increments only diagonal terms add up and hence $\mathbb{E}[S_{T\wedge n}^2]$ is increasing in n. We now argue that $\mathbb{E}[S_{T\wedge n}^2]$ increases to $\mathbb{E}[S_T^2]$. Since $\mathbb{E}[S_{T\wedge n}^2] = \mathbb{E}[T \wedge n]\sigma^2 < \sigma^2\mathbb{E}[T] < \infty$, we conclude that $S_{T\wedge n}$ is bounded in L^2. Therefore, by boundedness and orthogonality property, $S_{T\wedge n} \in L^2$ converges to some limit in L^2. However, we know that $S_{T\wedge n} \to S_T$ a.s. Therefore $\mathbb{E}[S_T^2] = \mathbb{E}[T]\sigma^2$. ■

References

[1] Richard Durrett. *Probability: theory and examples, 3rd edition.* Thomson Brooks/Cole, 2005.